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A selected tour of the theory of identification matrices ☆

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Abstract

A selected tour of the theory of identification matrices is offered here. We show that, among other things, shortest-path adjacency matrices are identification matrices for all simple graphs and adjacency matrices are identification matrices for all bipartite graphs. Additionally, we provide an improved proof that augmented adjacency matrices satisfying the circular 1's property are identification matrices. We also present a characterization of doubly convex bipartite graphs by identification matrices. Based on the theory of identification matrices, we describe an improved method for testing isomorphism between Γ circular arc graphs. The sequential algorithm can be implemented to run in $O(n^2)$ time and is optimal if the graphs are given as (augmented) adjacency matrices, so to speak.

Keywords: Identification matrices; Graph isomorphism; Bipartite graphs; Consecutive ones property

1. Introduction

The concept of identification matrices was formally introduced earlier this decade in studying the graph isomorphism problem, which was listed as an important open problem in [18] about a quarter of a century ago. Certain kinds of matrices have been shown to be identification matrices for some classes of graphs and the theory of identification matrices has helped us in designing efficient sequential and parallel algorithms. In this work, we present some new results about identification matrices and further enrich the theory. We show that shortest-path adjacency matrices are identification matrices for simple graphs, no matter whether the graphs are directed or undirected. We also show that adjacency matrices are identification matrices for all bipartite graphs. As part of this work, we give a characterization of doubly convex bipartite graphs by

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a kind of identification matrices and show that a graph is a doubly convex bipartite graph if and only if its adjacency matrix satisfies the consecutive 1's property. Also included is a refined proof that augmented adjacency matrices satisfying the circular 1's property are identification matrices. Then, as an application of the theory of identification matrices to the graph isomorphism problem, we present an improved method for testing isomorphism of Γ circular arc graphs. The parallel implementation reduces processor bound and work bound by a factor of n, where n is the number of vertices in a graph. The sequential implementation leads to an $O(n^2)$ time algorithm, which is optimal if the graphs are given as (augmented) adjacency matrices, so to speak.

In the next section, we present some definitions and briefly review some prior work that is used here. In Section 3, we show some important properties about identification matrices. In doing so, we also demonstrate certain fundamental techniques in proving various kinds of matrices are identification matrices. In Section 4, we use a specific example to show how the theory of identification matrices can help us to test graph isomorphism efficiently. Finally, in Section 5, we conclude the paper with some remarks.

2. Preliminaries

A *permutation matrix* is a square $(0,1)$-matrix with exactly one 1 in each of its rows and columns. The *identity matrix*, usually denoted by I, is a special permutation matrix, which has all its 1's on the main diagonal. Suppose P is a permutation matrix. Then PM is equivalent to permuting the rows of M, MP is equivalent to permuting the columns of M, and PMP^{t} is equivalent to permuting the rows and the corresponding columns of M, where P^{t} is the transpose of P. We can now see easily that the product of two permutation matrices is also a permutation matrix. Note that $PP^{\mathrm{t}} = I$, or equivalently, $P^{-1} = P^{\mathrm{t}}$. In other words, the inverse of a permutation matrix is its transpose.

Throughout this paper, we assume that all graphs are simple unweighted undirected graphs (i.e., unweighted undirected graphs without self loops or multiple edges), unless otherwise stated. A *clique* of a graph is a subset of the vertex set which induces a complete subgraph. A *maximal clique* is a clique not properly contained in another clique. Graph $G_1 = (V_1, E_1)$ is said to be *isomorphic* to graph $G_2 = (V_2, E_2)$ if there exists a one-to-one onto function f from V_1 to V_2 such that for any two distinct vertices, say v_i and v_j, in V_1, $(v_i, v_j) \in E_1$ if and only if $(f(v_i), f(v_j)) \in E_2$.

Let M_1 and M_2 be two matrices representing, respectively, two graphs G_1 and G_2 of a certain class $\mathscr{C}$, according to a certain relation $\mathscr{R}$. Suppose G_1 and G_2 are isomorphic if and only if there exist two permutation matrices P_1 and P_2 such that $M_1 = P_1 M_2 P_2$. Then M_1 and M_2 are said to be *identification matrices* for G_1 and G_2 of $\mathscr{C}$, with respect to $\mathscr{R}$.

Consider a simple example of identification matrices. Let M_1 and M_2 be the vertex vis-à-vis maximal clique incidence matrices for two simple graphs, G_1 and G_2, respectively. It can be readily seen that G_1 and G_2 are isomorphic if and only if there exist two permutation matrices P_1 and P_2 such that $M_1 = P_1M_2P_2$. Therefore, vertex vis-à-vis maximal clique incidence matrices are identification matrices for simple graphs. A detailed proof appeared in [5].

In the next section, we prove that some types of matrices are identification matrices for certain kinds of graphs. From the definition of identification matrices, we can see easily that a complete proof should show the following two parts: (1) if G_1 and G_2 are isomorphic, then there exist two permutation matrices P_1 and P_2 such that $M_1 = P_1M_2P_2$; (2) if there exist two permutation matrices P_1 and P_2 such that $M_1 = P_1M_2P_2$, then G_1 and G_2 are isomorphic, where M_1 and M_2 are two matrices representing, respectively, two graphs G_1 and G_2 of a certain class, according to a certain relation. As observed by a reviewer, part one is trivial, so it is omitted in the paper. In our proofs below, we also use an easily verifiable fact that $(AB)^t = B^tA^t$.

The following lemma can be derived from the definition of identification matrices.

Lemma 1. *Suppose M_1 and M_2 are identification matrices for graphs G_1 and G_2, with respect to a certain relation $\mathcal{R}$. Then two graphs are isomorphic if and only if there exists a permutation matrix P such that M_1 and M_2P have the same set of rows.*

Proof. ($\Rightarrow$) Suppose G_1 and G_2 are isomorphic. Then there exist two permutation matrices, say P_1 and P_2, such that $M_1 = P_1M_2P_2$, by the definition of identification matrices. It follows that M_1 and M_2P_2 have the same set of rows.

($\Leftarrow$) Suppose there exists a permutation matrix P such that M_1 and M_2P have the same set of rows. Then there exists another permutation matrix, say P_1, such that $M_1 = P_1M_2P$. It follows from the definition of identification matrix that G_1 and G_2 are isomorphic. □

Therefore, to test isomorphism of two graphs, given two identification matrices with respect to a certain relation, it suffices to test if, by permuting the columns, two (resulting) matrices can have the same set of rows.

We say that a $(0,1)$-matrix satisfies the *consecutive* 1*'s property for rows* if the columns of the matrix can be permuted such that the resulting matrix has consecutive 1's in each of its rows. A $(0,1)$-matrix is said to satisfy the *$p \times q$ consecutive* 1*'s property for rows* if there exists a submatrix of size $p \times q$ satisfying the consecutive 1's property for rows. A $(0,1)$-matrix is said to satisfy the *circular* 1*'s property for rows* if its columns can be permuted such that each row of the resulting matrix has circularly consecutive 1's. The consecutive (or the circular) 1's property for columns is defined analogously. The consecutive (or circular) 0's property is also defined analogously. If there is no mention of rows or columns, we assume that the property is for rows.

By definition, if a matrix satisfies the consecutive 1's property, then the matrix also satisfies the circular 1's property. So the circular 1's property can be regarded as a weakening or a relaxation of the consecutive 1's property.

An *interval graph* is an intersection graph of a set of intervals on a real line. The set of intervals is called the *intersection representation* of the interval graphs. If an interval graph has an intersection representation such that no interval is properly contained in another, then the graph is called a *proper interval graph*. A *circular arc graph* is an intersection graph of a set of circular arcs on a circle. The *proper circular arc graphs* are defined analogously.

A matrix is called an *augmented adjacency matrix* if it can be obtained from the adjacency matrix by adding 1's along the main diagonal. From the definitions we can readily see that if $M_1 = PM_2P^t$ for a permutation matrix P, then G_1 and G_2 are isomorphic, where M_1 and M_2 are both (augmented) adjacency matrices, representing G_1 and G_2, respectively.

A *shortest-path adjacency matrix* M for an n-vertex graph is a square matrix of order n whose element

$$m_{i,j} = \begin{cases} n & \text{if there is no path from } v_i \text{ to } v_j, \\ \text{length of the shortest path} & \\ \quad \text{from } v_i \text{ to } v_j & \text{otherwise} \end{cases}$$

for $0 < i,\ j \leqslant n$.

We say that a symmetric $(0,1)$-matrix has *circularly compatible* 1'*s* if the 1's in each column are circular and if, after inverting and/or cyclically permuting the order of the rows and the corresponding columns, the last 1 (in cyclically descending order) of the circular set in the second column is always at least as low as the last 1 of the circular set in the first column unless one of these columns is all 1's or all 0's.

Below is an example matrix with circularly compatible 1's.

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

More examples can be found in [6, 24].

The following result is observed in [24]. A complete proof is included in [6].

Lemma 2. *Suppose the n vertices of G are indexed so that the* 1'*s in each column of the augmented adjacency matrix M are circular. If this arrangement of M does not have circularly compatible* 1'*s, then M has the consecutive* 0'*s property and the vertices of G can be partitioned into two cliques.*

Corollary 1. *Suppose the* 1'*s in each column of an augmented adjacency matrix M are circular. If M does not have the consecutive* 0'*s property, then M has circularly compatible* 1'*s.*

Proof. Immediate from Lemma 2. □

The following matrix characterization of proper circular arc graphs is given in [24].

Theorem 1. *A graph is a proper circular arc graph if and only if there is an order of vertices such that the augmented adjacency matrix has circularly compatible* 1'*s.*

A complete proof of the characterization appears in [6]. Tucker [24] also showed that graphs whose augmented adjacency matrices satisfying the circular 1's property are circular arc graphs. Such graphs are called *Γ circular arc graphs* (also called *Tucker circular arc graphs*).

By definition, proper interval graphs are properly contained in interval graphs, which are in turn properly contained in circular arc graphs. For the relations between various subclasses of circular arc graphs and interval graphs, readers are referred to Chen [2, 6].

If the vertex set of a bipartite graph is ordered and partitioned into A and B such that the A vis-à-vis B incidence matrix has consecutive 1's in each of its rows, then the graph is called an *ordered convex bipartite graph*. If a graph can be turned into an ordered convex bipartite graph by ordering and partitioning its vertex set, then the graph is a *convex bipartite graph*. If the vertex set of a bipartite graph can be partitioned into A and B such that the A vis-à-vis B incidence matrix has the consecutive 1's property for both rows and columns, then the graph is called a *doubly convex bipartite graph*.

3. Properties of identification matrices

We are now ready to present the main results of this work. In our proofs below, we often use $\mathbf{1}$ to denote a matrix, consisting of 1's only, of proper size. For instance, $\mathbf{1}_s$ stands for an all-1 square matrix of order s. The subscript is omitted if no confusion arises. The meaning of $\mathbf{0}$ is analogous. A complement $(0,1)$-matrix, denoted by M^c, is a $(0,1)$-matrix that can be obtained from M by turning each of its 0-element into a 1-element and vice versa.

Theorem 2. *Shortest-path adjacency matrices are identification matrices for directed graphs.*

Proof. Suppose M_1 and M_2 are shortest-path adjacency matrices for two directed graphs G_1 and G_2, respectively, and M_2 can be transformed into M_1 by row and/or column permutation. Let P be the permutation matrix such that PM_2P^t has the same set of rows as M_1 does. Denote by R_i the ith row of M_1 and R_i' the ith row of PM_2P^t. Observe that R_i has exactly one 0-element, i.e., the ith element, and so does R_i', for $0<i\leqslant n$. It follows that $R_i=R_i'$, for $0<i\leqslant n$, and $PM_2P^t=M_1$. It can now be readily seen that for any two distinct vertices, say v_i and v_j, of G_1, (v_i,v_j) is an edge of G_1 if and only if $(f(v_i),f(v_j))$ is an edge of G_2, where f is a one-to-one onto function that corresponds

to P. Therefore, shortest-path adjacency matrices are identification matrices for directed graphs. □

Note that undirected graphs can be regarded as a special case of directed graphs with an undirected edge, say $\{v_i, v_j\}$, interpreted as two directed edges (v_i, v_j) and (v_j, v_i). Seidel [22] has shown that shortest-path adjacency matrices for undirected graphs can be constructed in $O(M(n)\log n)$ time, where $M(n)$ denotes the time necessary to multiply two matrices of order n and is $o(n^{2.376})$ from the work of Coppersmith and Winograd [12]. Shortest-path adjacency matrices can also be constructed by NC algorithms (see, e.g. [23]). It is interesting to note that isomorphism for directed graphs is polynomially equivalent to isomorphism for undirected graphs (see, e.g. [15]).

Lemma 3. *If augmented adjacency matrix M_1 can be turned into augmented adjacency matrix M_2 by row and column permutation, then the two corresponding graphs G_1 and G_2 have the same number of connected components.*

Proof. Without loss of generality, assume

$$M_1 = \begin{bmatrix} A_{1,1} & & & \\ & A_{2,2} & & \\ & & \ddots & \\ & & & A_{u,u} \end{bmatrix},$$

where u is the number of connected components in G_1 and $A_{i,i}$, of size $n_i \times n_i$, is the augmented adjacency matrix for the ith connected component, for $0 < i \leqslant u$. Suppose P is the permutation matrix which satisfies the property that PM_2P^{t} has the same set of rows as M_1 does. Let

$$PM_2P^{\mathrm{t}} = \begin{bmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,u} \\ B_{2,1} & B_{2,2} & \cdots & B_{2,u} \\ \vdots & \vdots & \vdots & \vdots \\ B_{u,1} & B_{u,2} & \cdots & B_{u,u} \end{bmatrix},$$

where $B_{i,i}$ is of the same size as $A_{i,i}$, for $0 < i \leqslant u$. Any row, say R_1, of PM_2P^{t} not in $[\,B_{1,1}\ B_{1,2}\ \cdots\ B_{1,u}\,]$ cannot equal a row, say R_2, in $[\,A_{1,1}\ \mathbf{0}\ \cdots\ \mathbf{0}\,]$, because there exists an $i > n_1$ such that $R_1(i) = 1$ but $R_2(j) = 0$ for $j > n_1$. It follows that $B_{1,i} = \mathbf{0}$, for $1 < i \leqslant u$, and $A_{1,1}$ and $B_{1,1}$ have the same set of rows.

For the same reason, $A_{i,i}$ and $B_{i,i}$ have the same set of rows for all i, and $B_{i,j} = \mathbf{0}$ for $i \neq j$. It follows that $u \leqslant v$, where v is the number of connected components in G_2. We can show analogously that $v \leqslant u$. Therefore, we conclude that G_1 and G_2 have the same number of connected components. □

Theorem 3. *Suppose $\mathscr{C}$ is such a class of graphs that if a graph is in $\mathscr{C}$, then each of its connected components is also in $\mathscr{C}$. If augmented adjacency matrices are identifi-*

cation matrices for connected graphs in $\mathscr{C}$, then they are also identification matrices for arbitrary graphs in $\mathscr{C}$.

Proof. Suppose augmented adjacency matrices are identification matrices for connected graphs in $\mathscr{C}$. Suppose M_1 and M_2 are, respectively, two augmented adjacency matrices for two graphs G_1 and G_2 in $\mathscr{C}$ and M_2 can be transformed into M_1 by row and column permutation. Without loss of generality, assume

$$M_1 = \begin{bmatrix} A_{1,1} & & & \\ & A_{2,2} & & \\ & & \ddots & \\ & & & A_{k,k} \end{bmatrix},$$

where k is the number of the connected components in G_1, and $A_{i,i}$, of size $n_i \times n_i$, is the augmented adjacency matrix for the ith connected component, for $0 < i \leqslant k$. Suppose P is the permutation matrix which satisfies the property that PM_2P^{t} has the same set of rows as M_1 does. We can then easily see from the proof of Lemma 3 that there exist $B_{1,1}$, $B_{2,2}, \ldots, B_{k,k}$ such that

$$PM_2P^{\mathrm{t}} = \begin{bmatrix} B_{1,1} & & & \\ & B_{2,2} & & \\ & & \ddots & \\ & & & B_{k,k} \end{bmatrix},$$

where $B_{i,i}$ has the same size and the same set of rows as $A_{i,i}$ does and is the augmented adjacency matrix for the ith connected component in G_2, for $0 < i \leqslant k$. Since the connected components of G_1 and G_2 are also in $\mathscr{C}$ and augmented adjacency matrices are identification matrices for connected graphs in $\mathscr{C}$, it follows that there exist permutation matrices P_1, $P_2, \ldots, P_k$ such that $P_iB_{i,i}P_i^{\mathrm{t}} = A_{i,i}$, for $0 < i \leqslant k$. Let

$$P' = \begin{bmatrix} P_1 & & & \\ & P_2 & & \\ & & \ddots & \\ & & & P_k \end{bmatrix}.$$

Then

$$\begin{aligned}(P'P)M_2(P'P)^{\mathrm{t}} &= P'PM_2P^{\mathrm{t}}(P')^{\mathrm{t}} \\ &= \begin{bmatrix} P_1 & & & \\ & P_2 & & \\ & & \ddots & \\ & & & P_k \end{bmatrix}\end{aligned}$$

$$
\begin{bmatrix} B_{1,1} & & & \\ & B_{2,2} & & \\ & & \ddots & \\ & & & B_{k,k} \end{bmatrix}
\begin{bmatrix} P_1^{\mathrm{t}} & & & \\ & P_2^{\mathrm{t}} & & \\ & & \ddots & \\ & & & P_k^{\mathrm{t}} \end{bmatrix}
$$

$$
= \begin{bmatrix} A_{1,1} & & & \\ & A_{2,2} & & \\ & & \ddots & \\ & & & A_{k,k} \end{bmatrix}
$$

$$
= M_1.
$$

This completes the proof. □

Theorem 4. *Adjacency matrices are identification matrices for the graphs in class $\mathscr{C}$ if and only if augmented adjacency matrices are identification matrices for the complements of the graphs in class $\mathscr{C}$.*

Proof. ($\Leftarrow$) Assume M_1 and M_2 are, respectively, adjacency matrices for G_1 and G_2 in $\mathscr{C}$, and M_2 can be turned into M_1 by row and column permutation. Then M_1^c and M_2^c are, respectively, augmented adjacency matrices for G_1^c and G_2^c, and M_2^c can be turned into M_1^c by row and column permutation. Since augmented adjacency matrices are identification matrices for the complements of the graphs in $\mathscr{C}$, it follows that there exists a permutation matrix P such that $M_1^c = PM_2^cP^{\mathrm{t}}$. Consequently, $M_1 = PM_2P^{\mathrm{t}}$. Therefore, adjacency matrices are identification matrices for the graphs in $\mathscr{C}$.

($\Rightarrow$) Analogous. □

Theorem 5. *Augmented adjacency matrices are identification matrices for graphs that can be partitioned into two cliques.*

Proof. Suppose M_1 and M_2 are augmented adjacency matrices for two graphs that can be partitioned into two cliques, and M_2 can be turned into M_1 by permutation of rows and columns. Without loss of generality, assume

$$
M_1 = \begin{bmatrix} \mathbf{1}_k & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1}_u & A \\ \mathbf{1} & A^{\mathrm{t}} & \mathbf{1} \end{bmatrix},
$$

where A does not contain any all-1 row or column, and $k \geqslant 0$. Let P be the permutation matrix such that PM_2P^{t} has the same set of rows as M_1 does. So PM_2P^{t} also has k all-1 rows. Since all rows of M_1 begin with at least k 1's, it follows that all rows of PM_2P^{t} also begin with at least k 1's. Note that PM_2P^{t} is symmetric. We can now see easily that the first k rows (and only the first k rows) of PM_2P^{t} are all 1's. Let v be the maximum integer such that

$$PM_2P^{\mathrm{t}} = \begin{bmatrix} \mathbf{1}_k & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1}_v & B \\ \mathbf{1} & B^{\mathrm{t}} & \mathbf{1} \end{bmatrix},$$

for a matrix B that does not contain any all-1 row. Below we shall show that $u = v$.

Assume $u < v$. Then M_1 has exactly $(u+k)$ rows that begin with at least $(u+k)$ 1's whereas PM_2P^{t} has at least $(v+k) > (u+k)$ rows that begin with at least $(u+k)$ 1's, which contradicts the assumption that M_1 and PM_2P^{t} have the same set of rows. It follows that $u \not< v$. We can show analogously that $u \not> v$. We therefore conclude that $u = v$. We can now see easily that A and B have the same set of rows. So there exists a permutation matrix P_1 such that $A = P_1B$. Let

$$P_2 = \begin{bmatrix} I_1 & & \\ & P_1 & \\ & & I_2 \end{bmatrix},$$

where I_1 and I_2 are identity matrices of orders k and $(n-k-v)$, respectively. Then

$$\begin{aligned}(P_2P)M_2(P_2P)^{\mathrm{t}} &= P_2PM_2P^{\mathrm{t}}P_2^{\mathrm{t}} \\ &= \begin{bmatrix} \mathbf{1}_k & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1}_v & P_1B \\ \mathbf{1} & B^{\mathrm{t}}P_1^{\mathrm{t}} & \mathbf{1} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{1}_k & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1}_v & A \\ \mathbf{1} & A^{\mathrm{t}} & \mathbf{1} \end{bmatrix} \\ &= M_1.\end{aligned}$$

Therefore, we conclude that augmented adjacency matrices are identification matrices for graphs that can be partitioned into two cliques. □

Corollary 2. *Augmented adjacency matrices are identification matrices for graphs whose connected components can each be partitioned into two cliques.*

Proof. By Theorems 3 and 5. □

Corollary 3. *Adjacency matrices are identification matrices for bipartite graphs.*

Proof. By Theorems 4 and 5. □

Note that isomorphism for arbitrary graphs is computationally equivalent to isomorphism for bipartite graphs since the former can be tested as efficiently as the latter by both sequential and parallel algorithms (see, e.g. [10]).

In the following, we shall use $M[r_1 : r_2, c_1 : c_2]$ to denote the submatrix composed of row r_1 through row r_2 and column c_1 through column c_2 of matrix M.

Lemma 4. *Suppose M is an adjacency matrix with consecutive 1's in each row and the graph is connected. Then $M = \begin{bmatrix} \mathbf{0} & A \\ A^{\mathrm{t}} & \mathbf{0} \end{bmatrix}$, for a matrix A, and the graph is a doubly convex bipartite graph.*

Proof. Note that the first element of row 1 of M is 0, since M is an adjacency matrix. Let k be the largest integer satisfying the property that the first k elements of row k of M are all 0's, for $0<k\leqslant n$, where n is the number of rows in M. We claim that $M[1:k, 1:k]=\mathbf{0}$. The claim can be easily proved by contradiction as follows. Assume there exist such i and j that $0<i,\ j<k$ and $m_{i,j}=1$. Then v_i and v_j cannot reach v_s for $k\leqslant s\leqslant n$, since M is symmetric and has consecutive 1's. This contradicts the given fact that the graph is connected. We therefore conclude that $M[1:k, 1:k]=\mathbf{0}$. The choice of k implies that at least one of the first k elements of row $(k+1)$ is 1. Since the 1's in each row are consecutive and $m_{k+1,k+1}=0$, it follows that $m_{k+1,j}=0$, for $k+1<j\leqslant n$. We can analogously prove that $M[k+1:n, k+1:n]=\mathbf{0}$. Therefore, $M = \begin{bmatrix} \mathbf{0} & A \\ A^{\mathrm{t}} & \mathbf{0} \end{bmatrix}$, for a matrix A. By definition, the graph is a doubly convex bipartite graph. $\square$

Theorem 6. *A graph is a doubly convex bipartite graph if and only if its adjacency matrix satisfies the consecutive 1's property.*

Proof. $(\Rightarrow)$ Immediate.

$(\Leftarrow)$ If an adjacency matrix, say M, satisfies the consecutive 1's property (for rows and columns), then there exists a permutation matrix, say P, such that PM has consecutive 1's in each of its columns. It follows that PMP^{t} has consecutive 1's in each of its rows and columns, since PMP^{t} is symmetric. Note that PMP^{t} is an adjacency matrix for the same graph with another vertex order. We can therefore assume, without loss of generality, that

$$M = \begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_k \end{bmatrix}$$

and M_i, $0<i\leqslant k$, is the adjacency matrix for the ith connected component of the graph and has consecutive 1's in each of its rows and columns. It follows easily from Lemma 4 that each connected component is a doubly convex bipartite graph. Therefore, the graph is a doubly convex bipartite graph. $\square$

We note that Roberts [21] has shown the following.

Theorem 7. *A graph is a proper interval graph if and only if its augmented adjacency matrix satisfies the consecutive 1's property.*

Corollary 4. *Adjacency matrices satisfying the consecutive* 1'*s property are identification matrices.*

Proof. If adjacency matrices satisfy the consecutive 1's property, then the graphs are doubly convex bipartite graphs by Theorem 6. Since adjacency matrices are identification matrices for bipartite graphs (recall Corollary 3), it then follows easily that adjacency matrices satisfying the consecutive 1's property are identification matrices. □

Corollary 5. *Augmented adjacency matrices satisfying the consecutive* 0'*s property are identification matrices.*

Proof. By Theorem 4 and Corollary 4. □

Lemma 5. *Suppose M is an augmented adjacency matrix with consecutive* 1'*s in each of its rows. Then, for any row except the top one, the* 1'*s neither begin nor end before the* 1'*s in the preceding row.*

Proof. Let R_i and R_{i+1} be two arbitrary consecutive rows of M. We prove below that the 1's in R_{i+1} do not begin before those in R_i. The other part of the proof is analogous and is therefore omitted.

Assume the contrary, i.e., the 1's in R_{i+1} begin before those in R_i. Then there exists such a j that $m_{i+1,j}=1$ and $m_{i,j}=0$ and $j<i$. Note that M is symmetric and $m_{j,j}=1$, since M is an augmented adjacency matrix. It follows that the 1's in column j are not consecutive, which contradicts the fact that the 1's in each row and column are consecutive. □

Theorem 8. *Augmented adjacency matrices satisfying the consecutive* 1'*s property are identification matrices.*

Proof. Suppose M_1 and M_2 are two augmented adjacency matrices, M_1 has consecutive 1's in each of its rows, and M_2 can be transformed into M_1 by row and column permutation. Then there exists a permutation matrix P such that PM_2P^{t} has the same set of rows as M_1 does. It follows that PM_2P^{t} also has consecutive 1's in each of its rows. We then use a pair of integers to represent a row in M_1 or M_2, with the first and the second integers corresponding to the column indices of the first and the last 1's, respectively. Denote by $(a_{1,i}, b_{1,i})$ the ith row of M_1, and $(a_{2,i}, b_{2,i})$ the ith row of PM_2P^{t}, for $0<i\leqslant n$. Obviously, $a_{i,j}\leqslant b_{i,j}$, for $0<i\leqslant 2$ and $0<j\leqslant n$. It follows from Lemma 5 that $a_{i,j}\leqslant a_{i,j+1}$ and $b_{i,j}\leqslant b_{i,j+1}$, for $0<i\leqslant 2$ and $0<j<n$. Note that M_1 and PM_2P^{t} have the same set of rows. It follows that $a_{1,i}=a_{2,i}$ and $b_{1,i}=b_{2,i}$, for $0<i\leqslant n$. Therefore, augmented adjacency matrices satisfying the consecutive 1's property are identification matrices. □

Using some additional ideas, we can obtain some stronger results under weaker conditions.

Theorem 9. *Augmented adjacency matrices satisfying the circular* 1'*s property are identification matrices.*

Proof. Suppose that M_1 and M_2 are two augmented adjacency matrices, M_1 has circular 1's in each of its rows, and M_2 can be transformed into M_1 by permuting its rows and columns.

Case 1: M_1 has the consecutive 0's property. By Corollary 5, augmented adjacency matrices satisfying the consecutive 0's property are identification matrices.

Case 2: M_1 does not have the consecutive 0's property. Note that in this case M_1 cannot contain an all-1 column. By Corollary 1, M_1 has circularly compatible 1's. We claim that M_1 can be written as

$$\begin{bmatrix} \mathbf{1}_u & A & B \\ A^{\mathrm{t}} & C & D \\ B^{\mathrm{t}} & D^{\mathrm{t}} & \mathbf{1} \end{bmatrix},$$

where submatrix $[\,A^{\mathrm{t}}\;\; C\;\; D\,]$ is non-empty and its first and last columns do not contain any 1's.

We prove the claim as follows. Let r_1 be the index of the last row of M_1 that has 1's from the first entry to the entry on the main diagonal. Let r_2 be the index of the first row of M_1 that has 1's from the entry on the main diagonal to the last entry. Then the submatrices $M_1[1 : r_1, 1 : r_1]$ and $M_1[r_2 : n, r_2 : n]$ consist of 1's only, for M_1 is symmetric and has circularly compatible 1's. Recall that M_1 has the circular 1's property but not the consecutive 0's property. It follows that the submatrix $[\,A^{\mathrm{t}}\;\; C\;\; D\,]$ contains at least one row. Suppose a row, say R, of $[\,A^{\mathrm{t}}\;\; C\;\; D\,]$ begins with a 1. Then either the row has 1's from the first entry to the entry on the main diagonal or the row has 1's from the entry on the main diagonal to the last entry. This contradicts the fact that r_1 is the index of the last row of M_1 that has 1's from the first entry to the entry on the main diagonal and r_2 is the index of the first row of M_1 that has 1's from the entry on the main diagonal to the last entry. We therefore conclude that no row in $[\,A^{\mathrm{t}}\;\; C\;\; D\,]$ begins with a 1. We can show analogously that no row in $[\,A^{\mathrm{t}}\;\; C\;\; D\,]$ ends with a 1. This completes the proof of the claim.

For $0<i\leqslant n$, we use a pair of integers $(a_{1,i}, b_{1,i})$ to represent row i of M_1, with

$$a_{1,i} = \text{the column index of the first 1 in the circular set}$$

$$b_{1,i} = \begin{cases} \text{the column index of the last 1 in the circular set} & \text{if the 1's in the row are consecutive,} \\ n + \text{the column index of the last 1 in the circular set} & \text{otherwise.} \end{cases}$$

By definition, $a_{1,i} \leqslant b_{1,i}$, for $0 < i \leqslant n$. Let s be the index of the first row in M_1 with consecutive 1's. Obviously, $s \leqslant r_1 + 1$. Recall that M_1 has circularly compatible 1's. It is now easy to see that there exists an integer s $(1 < s \leqslant r_1 + 1)$ such that for $i = s, s+1, \ldots, n, \ldots, (s-3+n) \bmod n + 1$, $a_{1,i} \leqslant a_{1,i \bmod n+1}$ and $b_{1,i} \leqslant b_{1,i \bmod n+1}$. Let P be the permutation matrix such that PM_2P^{t} has the same set of rows as M_1. Since PM_2P^{t} has circular 1's and does not have the consecutive 0's property, it follows from Corollary 1 that PM_2P^{t} has circularly compatible 1's. So we can denote PM_2P^{t} by

$$\begin{bmatrix} \mathbf{1}_v & E & F \\ E^{\mathrm{t}} & G & H \\ F^{\mathrm{t}} & H^{\mathrm{t}} & \mathbf{1} \end{bmatrix},$$

where the first and the last columns of the submatrix $[\, E^{\mathrm{t}} \;\; G \;\; H \,]$ do not contain any 1's. In the same way as for M_1, we use a pair of integers $(a_{2,i}, b_{2,i})$ to represent row i of PM_2P^{t}, for $0 < i \leqslant n$. Let t be the first row in M_2 with consecutive 1's. Then, for $i = t, t+1, \ldots, n, \ldots, (t-3+n) \bmod n + 1$, $a_{2,i} \leqslant a_{2,i \bmod n+1}$ and $b_{2,i} \leqslant b_{2,i \bmod n+1}$. Since M_1 and PM_2P^{t} have the same set of rows, we conclude that $a_{1,(s+i-1) \bmod n+1} = a_{2,(t+i-1) \bmod n+1}$ and $b_{1,(s+i-1) \bmod n+1} = b_{2,(t+i-1) \bmod n+1}$, for $0 \leqslant i < n$. So $[\, A^{\mathrm{t}} \;\; C \;\; D \,] = [\, E^{\mathrm{t}} \;\; G \;\; H \,]$. To conclude $M_1 = PM_2P^{\mathrm{t}}$, it now suffices to show that $u = v$.

Suppose $u < v$. Then the last row R_n of M_1 equals a row, say R_i', in $[\, \mathbf{1}_v \;\; E \;\; F \,]$. Since the last column of D does not contain any 1's, it follows that the last row of D^{t} does not contain any 1's. Consider the $(u+1)$st element of R_n. It is in the last row of D^{t}, so it is a 0-entry. However, the $(u+1)$st element of R_i' is a 1-entry. Thus a contradiction is derived. It follows that $u \nless v$. Analogously, $u \ngtr v$. We therefore conclude that $u = v$ and $PM_2P^{\mathrm{t}} = M_1$.

This completes the proof that augmented adjacency matrices satisfying the circular 1's property are identification matrices. □

Before proving another result on identification matrices, we observe a few interesting properties of matrices.

Observation 1. *Suppose two $m \times n$ matrices M_1 and M_2 have the same set of rows. Then column i of M_1 contains the same number of* 1*'s as column i of M_2 does, for any i, $0 < i \leqslant n$.*

Observation 2. *Suppose two $m \times n$ matrices M_1 and M_2 have the same set of columns. Then row i of M_1 contains the same number of* 1*'s as row i of M_2 does, for any i, $0 < i \leqslant m$.*

Note that in the above observations, array element "1" can be replaced by another array element such as "0".

Theorem 10. *Augmented adjacency matrices of order n satisfying the $n \times (n-1)$ consecutive* 1*'s property are identification matrices.*

Proof. Suppose M_1 and M_2 are two augmented adjacency matrices. Assume, without loss of generality, that M_1 and M_2 have the same set of rows and

$$M_1 = \begin{bmatrix} A & B & C \\ B^t & 1 & D \\ C^t & D^t & E \end{bmatrix},$$

where

$$\begin{bmatrix} A & C \\ B^t & D \\ C^t & E \end{bmatrix}$$

has consecutive 1's in each of its rows, and $[\,B^t \;\; 1 \;\; D\,]$ is the pth row of M_1, i.e., $M_1[p:p,1:n]$, or simply $M_1[p,1:n]$. Also assume, without loss of generality, that each node has at least one neighbor, or equivalently, each row of M_1 or M_2 contains at least two 1's. Let

$$M_2 = \begin{bmatrix} F & G & H \\ G^t & 1 & J \\ H^t & J^t & K \end{bmatrix},$$

where $[\,G^t \;\; 1 \;\; J\,]$ is the pth row of M_2. Below we shall represent row i of M_1 by a triple $(s_{1,i}, e_{1,i}, b_{1,i})$, for $0<i\leqslant n$, as follows.

$$s_{1,i} = k \text{ satisfying } M_1[i,k]=1,\ k\neq p, \text{ and } M_1[i,j]=0 \text{ for } 0<j<k \text{ and } j\neq p,$$

$$e_{1,i} = k \text{ satisfying } M_1[i,k]=1,\ k\neq p, \text{ and } M_1[i,j]=0 \text{ for } k<j\leqslant n \text{ and } j\neq p,$$

$$b_{1,i} = \text{the } i\text{th element of } M_1[p,1:n].$$

Observe that if we replace the pth column of M_1 by an all-0 column, then $s_{1,i}$ and $e_{1,i}$ represent, respectively, the indices of the first and the last 1's in row i, for $0<i\leqslant n$. If we delete the pth column of M_1, then the 1's in each row are consecutive. By definition, $s_{1,i}\leqslant e_{1,i}$, for $0<i\leqslant n$. It is now easy to see that

$$b_{1,i} = \begin{cases} 1 & \text{if } s_{1,p}\leqslant i\leqslant e_{1,p} \text{ or } i=p, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\begin{bmatrix} A & C \\ C^t & E \end{bmatrix}$ is an augmented adjacency matrix with consecutive 1's in each of its rows, it then follows from Lemma 5 that $s_{1,i}\leqslant s_{1,j}$ and $e_{1,i}\leqslant e_{1,j}$, for $0<i\leqslant j\leqslant n$, $i\neq p$ and $j\neq p$. Since M_2 has the same set of rows as M_1 does, it follows that

$$\begin{bmatrix} F & H \\ G^t & J \\ H^t & K \end{bmatrix}$$

also has consecutive 1's in each of its rows. In an analogous way, we use $(s_{2,i}, e_{2,i}, b_{2,i})$ to represent row i of M_2, for $0<i\leqslant n$. For the same reason as above, $s_{2,i}\leqslant s_{2,j}$ and $e_{2,i}\leqslant e_{2,j}$, for $0<i\leqslant j\leqslant n$, $i\neq p$ and $j\neq p$.

Note that if $M_1[p,1:n]=M_2[p,1:n]$ (i.e., $[\,B^{\mathrm{t}}\ \ 1\ \ D\,]=[\,G^{\mathrm{t}}\ \ 1\ \ J\,]$), then

$$\begin{bmatrix} A & C \\ C^{\mathrm{t}} & E \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} F & H \\ H^{\mathrm{t}} & K \end{bmatrix}$$

have the same set of rows. Following the same arguments in the proof of Theorem 8, we can now easily show that if $M_1[p,1:n]=M_2[p,1:n]$, then $M_1=M_2$. So it now suffices to show that $M_1[p,1:n]=M_2[p,1:n]$. Below we prove the equality by contradiction.

Assume $M_1[p,1:n]\neq M_2[p,1:n]$, i.e., $(s_{1,p},e_{1,p},1)\neq(s_{2,p},e_{2,p},1)$. From Observation 2, we conclude that row i of M_1 contains the same number of 1's as that of M_2 for any i. It follows that $M_1[p,1:n]$ and $M_2[p,1:n]$ contain the same number of 1's. We can now see that $s_{1,p}\neq s_{2,p}$.

Case 1: $s_{1,p}<s_{2,p}$. Then $b_{1,s_{1,p}}=1$ and $b_{2,s_{1,p}}=0$, and $b_{1,i}=b_{2,i}=0$, for $0<i<s_{1,p}$ and $i\neq p$. We can now conclude that row $s_{1,p}$ of M_1 does not equal row $s_{1,p}$ of M_2, and that row i of M_1 equals that of M_2, for $0<i<s_{1,p}$ and $i\neq p$, because those rows have the lexicographically smallest triples among the rows whose pth element is 0 and M_1 and M_2 have the same set of rows. Among the rest of the rows, there exist u and v such that row u of M_1 equals row $s_{1,p}$ of M_2 and row v of M_2 equals row $s_{1,p}$ of M_1, i.e., $(s_{2,s_{1,p}},e_{2,s_{1,p}},0)=(s_{1,u},e_{1,u},0)$ and $(s_{1,s_{1,p}},e_{1,s_{1,p}},1)=(s_{2,v},e_{2,v},1)$. Note that $u\geqslant s_{1,p}$ or $u=p$, and $v\geqslant s_{1,p}$ or $v=p$. By definition, $b_{1,p}=1$. Since $b_{1,u}=0$, it follows that $u\neq p$.

Case 1.1: $v\neq p$. Then

$$\begin{aligned} s_{1,s_{1,p}} &\leqslant s_{1,u} \text{ (recall } p\neq s_{1,p}\leqslant u\neq p, \text{ and } s_{1,i}\leqslant s_{1,j} \text{ for } p\neq i\leqslant j\neq p) \\ &= s_{2,s_{1,p}} \text{ (recall row } u \text{ of } M_1 \text{ equals row } s_{1,p} \text{ of } M_2) \\ &\leqslant s_{2,v} \text{ (recall } p\neq s_{1,p}\leqslant v\neq p, \text{ and } s_{2,i}\leqslant s_{2,j} \text{ for } p\neq i\leqslant j\neq p) \\ &= s_{1,s_{1,p}} \text{ (recall row } v \text{ of } M_2 \text{ equals row } s_{1,p} \text{ of } M_1). \end{aligned}$$

It follows that $s_{1,s_{1,p}}=s_{2,v}=s_{2,s_{1,p}}=s_{1,u}$. Analogously, $e_{1,s_{1,p}}=e_{2,v}=e_{2,s_{1,p}}=e_{1,u}$. Then row $s_{1,p}$ of M_1 (i.e., $(s_{1,s_{1,p}},e_{1,s_{1,p}},1)$) contains more 1's than row $s_{1,p}$ of M_2 (i.e., $(s_{2,s_{1,p}},e_{2,s_{1,p}},0)$) does. Thus we have derived a contradiction.

Case 1.2: $v=p$. Then

$$\begin{aligned} M_2[s_{1,p},p] &= M_2[p,s_{1,p}] \text{ (recall } M_2 \text{ is symmetric)} \\ &= M_2[v,s_{1,p}] \text{ (recall } v=p) \\ &= M_1[s_{1,p},s_{1,p}] \text{ (recall row } v \text{ of } M_2 \text{ equals row } s_{1,p} \text{ of } M_1) \\ &= 1 \text{ (recall } M_1 \text{ has all 1's on its main diagonal).} \end{aligned}$$

It follows that $s_{2,p}\leqslant s_{1,p}$, which contradicts the assumption that $s_{1,p}<s_{2,p}$.

Case 2: $s_{1,p}>s_{2,p}$. Analogous to case 1.

We therefore conclude that $M_1[p,1:n]=M_2[p,1:n]$ and $M_1=M_2$. Consequently, augmented adjacency matrices of order n satisfying the $n\times(n-1)$ consecutive 1's property are identification matrices. □

4. An application

Based on the theory of identification matrices, Chen [5] devised an NC isomorphism testing algorithm for graphs represented by a kind of identification matrices satisfying the consecutive 1's property. By invoking this procedure n times, where n is the number of columns in a matrix, Chen [5] then showed that isomorphism for graphs represented by a kind of identification matrices satisfying the circular 1's property is also in NC. Nevertheless, the processor bound and the work bound of the procedure on matrices satisfying the circular 1's property are by a factor of n greater than those of the procedure on matrices satisfying the consecutive 1's property. Below we shall show that for graphs whose augmented adjacency matrices satisfying the circular 1's property (i.e., Γ circular arc graphs), isomorphism can be tested by invoking a parallel algorithm on identification matrices satisfying the consecutive 1's property only a constant number of times, thus reducing the processor bound and also the work bound by a factor of n. The corresponding sequential algorithm can be implemented to run in $O(n^2)$ time.

Theorem 11. *Suppose M is an augmented adjacency matrix for a connected graph. If M has circularly compatible* 1*'s and no all-*1 *rows, then M is unique up to cyclical permutation and inversion of its rows and the corresponding columns.*

Proof. We claim that M can be written as

$$\begin{bmatrix} \mathbf{1}_u & A & B \\ A^{\mathrm{t}} & C & D \\ B^{\mathrm{t}} & D^{\mathrm{t}} & \mathbf{1} \end{bmatrix},$$

where submatrix $[A^{\mathrm{t}}\ C\ D]$, if not empty, contains 0's only in its first and last columns. We prove the claim as follows. Let r_1 be the index of the last row of M that has 1's from the first entry to the entry on the main diagonal. Let r_2 be the index of the first row of M that has 1's from the entry on the main diagonal to the last entry. Then the submatrices $M(1:r_1, 1:r_1)$ and $M(r_2:n, r_2, n)$ consist of 1's only and $r_1 < r_2$, for M is symmetric and has circularly compatible 1's but no all-1 rows. Note that $r_1 + 1 = r_2$ if and only if submatrix $[A^{\mathrm{t}}\ C\ D]$ is empty. Suppose the submatrix is not empty and has a row that begins with a 1. Then either the row has 1's from the first entry to the entry on the main diagonal or the row has 1's from the entry on the main diagonal to the last entry. This contradicts the fact that r_1 is the index of the last row of M that has 1's from the first entry to the entry on the main diagonal and r_2 is the index of the first row of M that has 1's from the entry on the main diagonal to the last entry. We therefore conclude that no row in $[A^{\mathrm{t}}\ C\ D]$ begins with a 1. We can show analogously that if $[A^{\mathrm{t}}\ C\ D]$ is not empty, then the submatrix does not contain a row that ends with a 1. This completes the proof of the claim. Now, let's use a pair of integers (a_i, b_i) to represent row i of M, for $0 < i \leqslant n$, in the same way as in the proof of Theorem 9. Since M has circularly compatible 1's and no all-1 rows, it follows that there exists an integer s such that for $i = s, s+1, \ldots, n, \ldots, (s-3+n) \bmod n + 1$,

$a_i \leqslant a_{i \bmod n+1}$ and $b_i \leqslant b_{i \bmod n+1}$. So, if M contains identical rows, they must appear in circularly consecutive order. We then delete all row and column duplicates. The resulting matrix is still denoted by M. Now, it suffices to show that M is unique up to cyclical permutation and inversion of its rows and the corresponding columns. Note that, by Theorem 1, the graph represented by M is a proper circular arc graph. Suppose A is an arbitrary set of proper circular arcs that represents the graph. Let $A_0, A_1, \ldots, A_{n-1}$ be the arcs in A in sorted order. Without loss of generality, assume row 1 of M corresponds to A_i, for an i. Since M has no all-1 rows and the graph is connected, we conclude that row 2 of M corresponds to either $A_{(i+1) \bmod n}$ or $A_{(i-1+n) \bmod n}$. Without loss of generality, assume row 2 corresponds to $A_{(i+1) \bmod n}$. Then row k corresponds to $A_{(i+k-1) \bmod n}$, for $0 < k \leqslant n$. We can therefore see that the first two rows of M uniquely determine M. Since cyclical permutation and inversion of the rows and the corresponding columns have no effect on circularly compatible 1's, we now conclude that M is unique up to cyclical permutation and inversion of its rows and the corresponding columns. $\square$

Below we describe an improved approach for testing isomorphism between Γ circular arc graphs. The input is two Γ circular arc graphs represented by two augmented adjacency matrices M_1 and M_2 satisfying the circular 1's property.

1. Check if the two matrices contain the same number of all-1 rows. If not, we conclude immediately that the two graphs are not isomorphic and stop. Otherwise, we update M_1 and M_2 by deleting all their all-1 rows and columns.
2. Check if M_1 and M_2 satisfy the consecutive 0's property. If so, the complement graphs are doubly convex bipartite graphs by Theorem 6, so we use a parallel algorithm for doubly convex bipartite graphs to test isomorphism and then stop. (Note that two graphs are isomorphic if and only if their complements are isomorphic.)
3. Since M_1 and M_2 do not satisfy the consecutive 0's property, it then follows from Corollary 1 and Theorem 1 that the graphs are proper circular arc graphs. Check if the graphs are connected. If not, then the graphs are proper interval graphs, so we use a parallel algorithm for proper interval graphs to test isomorphism and then stop.
4. Transform the matrices into ones with circular (compatible) 1's, and then decide isomorphism. (Note that matrices that have circular 1's but do not satisfy the consecutive 0's property also have circularly compatible 1's by Corollary 1.)

Note that adjacency matrices for doubly convex bipartite graphs satisfy the consecutive 1's property (recall Theorem 6) and are identification matrices (recall Corollary 3), and augmented adjacency matrices for proper interval graphs satisfy the consecutive 1's property (recall Theorem 7) and are identification matrices (recall Theorem 9). Isomorphism for these graphs can be tested by invoking an algorithm that tests isomorphism for graphs represented by identification matrices satisfying the consecutive 1's property, which can be derived from the algorithm that tests isomorphism for labelled PQ-trees [20]. Isomorphism for proper interval graphs can also be tested in another way (see, e.g. [11, 13]). The consecutive and circular 1's or 0's properties can also be recognized in $O(n^2)$ sequential time [1, 16]. Testing isomorphism for connected proper circular arc graphs in Step 4 of the preceding procedure can be done based on Theorem 11. By cyclical permutation and inversion of the rows and the corresponding columns of a matrix, we can derive $O(n)$ matrices. Each row of a matrix with circular 1's can be represented by a pair of integers. So a matrix can be represented by a sequence of n pairs of integers. Checking whether two matrices are identical can therefore be done in $O(n)$ time. To test isomorphism of two graphs, we need check $O(n)$ pairs of matrices. It is now easy to conclude the following.

Theorem 12. *Isomorphism for Γ circular arc graphs can be tested in* $O(n^2)$ *time.*

The procedure is obviously optimal if the graphs are given as (augmented) adjacency matrices. Note that for arbitrary circular arc graphs, efficient sequential isomorphism testing algorithms have been designed (see, e.g. [17, 25]), though they take more time in the worst case.

On a Common CRCW PRAM, Chen [5] showed previously that isomorphism for Γ circular arc graphs can be decided in $O(\log^2 n)$ time with $O(n^4)$ Common CRCW PRAM processors. With some previously published procedures and frequently used techniques (see, e.g. [5, 6, 9, 19]), we can easily show that the preceding procedure runs in $O(\log^2 n)$ time with $O(n^3)$ Common CRCW PRAM processors. In fact, our new isomorphism testing algorithm can be implemented to run more efficiently, since the dominating steps published previously can be improved (see, e.g. [8]); but that's beyond the scope of this work. Our main purpose of treating the isomorphism for Γ circular arc graphs here is to give an example of applying the theory of identification matrices.

5. Concluding remarks

We have presented some fundamental properties of identification matrices. Although the theory of identification matrices was not formulated until this decade, identification matrices have been used for at least decades in applications such as isomorphism detection, characterization, and recognition of various classes of graphs including interval graphs (see, e.g. [14]) and proper circular arc graphs (see, e.g. [24]). Such kind of phenomena is not rare in the scientific world. Famous examples can be found in computer science as well as in some other fields. So far, the theory of identification matrices has helped us in understanding some graph problems and finding efficient solutions. We believe that the theory will further help us with our work in the future.

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